Representation Theory – Midterm 2026/27 Semester I

Each question has 5 **marks** – **Total** $6 \times 5 = 30$ **Marks** – **Time 180 minutes**

Any result proved in the class can be used without a proof after explicitly stating it, unless the question itself asks for a proof of that result. You may NOT use a result from the exercise sheet without proof.

Solve all of (1) – (3)

(1) Let G be a locally compact group with a right Haar measure μ. Show that Δ^{-1} is the Jacobian of the change of variable for $x \mapsto x^{-1}$, that is, show that

$$\int_G f(x)\, d\mu(x) = \int_G f(x^{-1})\Delta(x^{-1})\, d\mu(x)$$

for $f \in C_c(G)$.

(2) Let T be a positive bounded self-adjoint compact operator on a separable Hilbert space.

(a) Show that

$$\|T\| = \sup\{|\langle Tv, v\rangle| : \|v\| = 1\}$$

is an eigenvalue of T. You do NOT need to prove the above equality.

(b) Using the Spectral Theorem show that T can have only countable many positive eigenvalues.

(3) Assume that any cyclic representation of a compact group has a countable orthogonal decomposition into finite dimensional representations. Show that any unitary representation on a separable Hilbert space of that group can be orthogonally decomposed into countably many irreducible representations.

Solve any 3 of (4) – (7)

(4) Prove that any sub-representation of $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ on $L^2(\mathbb{T})$ is of the form

$$E_X = \oplus_{n \in X} \mathbb{C} e_n,$$

for some subset $X \subseteq \mathbb{Z}$. Here, as usual, $e_n(z) = \exp(2\pi i n z)$.

(5) Let G_i be compact groups. Using Peter–Weyl theorem and the character theory show that there is a bijection between $\widehat{G}_1 \times \widehat{G}_2$ and $\widehat{G_1 \times G_2}$.

(6) Let G be a compact group and $\pi \in \widehat{G}$.

(a) Show that

$$\dim \operatorname{Hom}_G\left(\pi^{\oplus m}, \pi^{\oplus n}\right) = mn,$$

for $m, n \in \mathbb{Z}_{\geq 0}$.

(b) Show that

$$\dim \operatorname{Hom}_G\left(\pi \otimes \pi^*, \mathbb{C}\right) = 1,$$

where π^* denote the dual representation of π. Find an explicit non-zero element in the above Hom-space.

(7) Let G be a finite group.

(a) Show that $\widehat{G}$ has the same cardinality as the conjugacy classes of G.

(b) Show that

$$\sum_{\pi \in \widehat{G}} \chi_\pi(x)\overline{\chi_\pi(y)} = \begin{cases} |Z_G(x)|, & y \text{ conjugate to } x; \\ 0. & \text{otherwise.} \end{cases}$$

where $Z_G(x)$ denotes the centralizer of x.